

A Maschke type theorem for relative Hom-Hopf modules

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Abstract In this paper, we prove a Maschke type theorem for the category of relative Hom-Hopf modules. In fact, we give necessary and sufficient conditions for the functor that forgets the (H, α) -coaction to be separable. This leads to a generalized notion of integrals.

Keywords Monoidal Hom-Hopf algebra; separable functors; Maschke type theorem; total integral; relative Hom-Hopf module.

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1 Introduction

The present paper investigates variations on the theme of Hom-algebras, a topic which has recently received much attention from various researchers. The study of Hom-associative algebras originates with work by Hartwig, Larsson and Silvestrov in the Lie case [9], where a notion of Hom-Lie algebra was introduced in the context of studying deformations of Witt and Virasoro algebras. Later, it was extended to the associative case by Makhlof and Silvestrov in [10]. Now the associativity is replaced by Hom-associativity $\alpha(a)(bc) = (ab)\alpha(c)$. Hom-coassociativity for a Hom-coalgebra can be considered in a similar way, see [10]. Caenepeel etc. [1] studied Hom-structures from the point of view of monoidal categories. This leads to the natural definition of monoidal Hom-algebras, Hom-coalgebras, etc. They constructed a symmetric monoidal category, and then introduced monoidal Hom-algebras, Hom-coalgebras, etc. as algebras, coalgebras, etc. in this monoidal category.

The notion of a relative (H, B) -Hopf module, where H is a Hopf algebra over a field k and B is a right coideal subalgebra of H , was introduced and studied by Takeuchi in [12]. Later, in [3](see also [4]), Doi noted that the notion of an (H, B) -Hopf module works well if B is a right H -comodule algebra, Using this module, he proved that the existence of a total integral $\phi : H \rightarrow B$ is equivalent to B being a relative injective H -comodule, and it is also equivalent to any (H, B) -Hopf module M being a relative injective H -comodule in [5].

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Also, in [5], using a commutative assumption for H , he deduced a version of the Maschke type theorem for (H, B) -Hopf modules which states that every exact sequence of (H, B) -Hopf modules which splits B -linearly, also splits (H, B) -linearly. Afterwards, Doi proved in [5] that the commutative condition can be removed and replaced by some technical conditions involving the center of B . Caenepeel etc. [2] proved a Maschke type theorem for the category of relative Hopf modules. In fact, they gave necessary and sufficient conditions for the functor that forgets the H -coaction to be separable. This leads to a generalized notion of integrals of Doi [5].

In this paper we study the generalization of the previous results to the Hom-Hopf algebras. In Sec.3, we introduce the notion of a relative Hom-Hopf module and prove that the functor F from the category of relative Hom-Hopf modules to the category of right (A, β) -Hom-modules has a right adjoint (see Proposition 3.3). In Sec.4, we introduce the notion of total integrals for Hom-comodule algebras, which is an effective tool for investigating properties of relative Hom-Hopf modules. As an important application, we investigate the injectivity of relative Hom-Hopf modules (see Proposition 4.3), which generalizes the main result in [3]. In Sec.5, we obtain the main result of this paper. we give necessary and sufficient conditions for the functor that forgets the (H, α) -coaction to be separable (see Theorem 5.2), we prove a Maschke type theorem for the category of relative Hom-Hopf modules as an application. In fact, let (A, β) be a right (H, α) -Hom-comodule algebra with a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$. If $\phi : (H, \alpha) \rightarrow (Z(A), \beta)$ (the center of A) is a multiplication map, then every short exact sequence of relative Hom-Hopf modules

$$0 \longrightarrow (M, \mu) \xrightarrow{f} (N, \nu) \xrightarrow{g} (P, \pi) \longrightarrow 0$$

which splits as a sequence of (A, β) -Hom-modules also splits as a sequence of relative Hom-Hopf modules.

2 Preliminaries

Throughout this paper we work over a commutative ring k , we recall from [1] some information about Hom-structures which are needed in what follows.

Let \mathcal{C} be a category. We introduce a new category $\widetilde{\mathcal{H}}(\mathcal{C})$ as follows: objects are couples (M, μ) , with $M \in \mathcal{C}$ and $\mu \in \text{Aut}_{\mathcal{C}}(M)$. A morphism $f : (M, \mu) \rightarrow (N, \nu)$ is a morphism $f : M \rightarrow N$ in \mathcal{C} such that $\nu \circ f = f \circ \mu$.

Let \mathcal{M}_k denotes the category of k -modules. $\mathcal{H}(\mathcal{M}_k)$ will be called the Hom-category associated to \mathcal{M}_k . If $(M, \mu) \in \mathcal{M}_k$, then $\mu : M \rightarrow M$ is obviously a morphism in $\mathcal{H}(\mathcal{M}_k)$. It is easy to show that $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (I, I), \tilde{a}, \tilde{l}, \tilde{r})$ is a monoidal category by Proposition 1.1 in [1]: the tensor product of (M, μ) and (N, ν) in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is given by the formula $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$.

Assume that $(M, \mu), (N, \nu), (P, \pi) \in \mathcal{H}(\mathcal{M}_k)$. The associativity and unit constraints are given by the formulas

$$\begin{aligned} \tilde{a}_{M, N, P}((m \otimes n) \otimes p) &= \mu(m) \otimes (n \otimes \pi^{-1}(p)), \\ \tilde{l}_M(x \otimes m) &= \tilde{r}_M(m \otimes x) = x\mu(m). \end{aligned}$$

An algebra in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ will be called a monoidal Hom-algebra.

Definition 2.1. A monoidal Hom-algebra is an object is a triple $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $m_A : A \otimes A \rightarrow A$ and an element $1_A \in A$ such that

$$\begin{aligned}\alpha(ab) &= \alpha(a)\alpha(b); \quad \alpha(1_A) = 1_A, \\ \alpha(a)(bc) &= (ab)\alpha(c); \quad a1_A = 1_Aa = \alpha(a),\end{aligned}$$

for all $a, b, c \in A$. Here we use the notation $m_A(a \otimes b) = ab$.

Definition 2.2. A monoidal Hom-coalgebra is an object $(C, \gamma) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with k -linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\gamma : C \rightarrow C$ such that

$$\Delta(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}); \quad \varepsilon(\gamma(c)) = \varepsilon(c),$$

and

$$\gamma^{-1}(c_{(1)}) \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes \gamma(c_{(2)}), \quad \varepsilon(c_{(1)})c_{(2)} = \varepsilon(c_{(2)})c_{(1)} = \gamma^{-1}(c)$$

for all $c \in C$.

Definition 2.3. A monoidal Hom-bialgebra $H = (H, \alpha, m, \eta, \Delta, \varepsilon)$ is a bialgebra in the symmetric monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$. This means that (H, α, m, η) is a Hom-algebra, (H, Δ, α) is a Hom-coalgebra and that Δ and ε are morphisms of Hom-algebras that is,

$$\begin{aligned}\Delta(ab) &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}; \quad \Delta(1_H) = 1_H \otimes 1_H, \\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b), \quad \varepsilon(1_H) = 1_H.\end{aligned}$$

Definition 2.4. A monoidal Hom-Hopf algebra is a monoidal Hom-bialgebra (H, α) together with a linear map $S : H \rightarrow H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that

$$S * I = I * S = \eta\varepsilon, \quad S\alpha = \alpha S.$$

Definition 2.5. Let (A, α) be a monoidal Hom-algebra. A right (A, α) -Hom-module is an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ consists of a k -module and a linear map $\mu : M \rightarrow M$ together with a morphism $\psi : M \otimes A \rightarrow M, \psi(m \cdot a) = m \cdot a$, in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that

$$(m \cdot a) \cdot \alpha(b) = \mu(m) \cdot (ab); \quad m \cdot 1_A = \mu(m),$$

for all $a \in A$ and $m \in M$. The fact that $\psi \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that

$$\mu(m \cdot a) = \mu(m) \cdot \alpha(a).$$

A morphism $f : (M, \mu) \rightarrow (N, \nu)$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called right A -linear if it preserves the A -action, that is, $f(m \cdot a) = f(m) \cdot a$. $\widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ will denote the category of right (A, α) -Hom-modules and A -linear morphisms.

Definition 2.6. Let (C, γ) be a monoidal Hom-coalgebra. A right (C, γ) -Hom-comodule is an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k -linear map $\rho_M : M \rightarrow M \otimes C$ notation $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that

$$m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})) = \mu^{-1}(m_{[0]}) \otimes \Delta_C(m_{[1]}); \quad m_{[0]}\varepsilon(m_{[1]}) = \mu^{-1}(m),$$

for all $m \in M$. The fact that $\rho_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ means that

$$\rho_M(\mu(m)) = \mu(m_{[0]}) \otimes \gamma(m_{[1]}).$$

Morphisms of right (C, γ) -Hom-comodule are defined in the obvious way. The category of right (C, γ) -Hom-comodules will be denoted by $\widetilde{\mathcal{H}}(\mathcal{M}_k)^C$.

3 Adjoint functor

Definition 3.1. Let (H, α) be a monoidal Hom-Hopf algebra. A monoidal Hom-algebra (A, β) is called a right (H, α) -Hom-comodule algebra if (A, β) is a right (H, α) Hom-comodule with coaction $\rho_A : A \rightarrow A \otimes H$, $\rho_A(a) = a_{[0]} \otimes a_{[1]}$ such that the following conditions satisfy,

$$\begin{aligned}\rho_A(ab) &= a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}, \\ \rho_A(1_A) &= 1_A \otimes 1_H.\end{aligned}$$

for all $a, b \in A$.

Definition 3.2. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra. A relative Hom-Hopf module (M, μ) is a right (A, β) -Hom-module which is also a right (H, α) -Hom-comodule with the coaction structure $\rho_M : M \rightarrow M \otimes H$ defined by $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ such that the following compatible condition holds: for all $m \in M$ and $a \in A$,

$$\rho_M(ma) = m_{[0]} \cdot a_{[0]} \otimes m_{[1]}a_{[1]}.$$

A morphism between two right relative Hom-Hopf modules is a k -linear map which is a morphism in the categories $\widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ and $\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_A$ at the same time. $\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_A$ will denote the category of right relative Hom-Hopf modules and morphisms between them.

Proposition 3.3. The forgetful functor $F : \widetilde{\mathcal{H}}(\mathcal{M}_k)^H_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$ has a right adjoint $G : \widetilde{\mathcal{H}}(\mathcal{M}_k)_A \rightarrow \widetilde{\mathcal{H}}(\mathcal{M}_k)^H_A$. G is defined by

$$G(M) = M \otimes H,$$

with structure maps

$$(m \otimes h) \cdot a = m \cdot a_{[0]} \otimes ha_{[1]},$$

$$\rho_{G(M)}(m \otimes h) = (\mu^{-1}(m) \otimes h_{(1)}) \otimes \alpha(h_{(2)}),$$

for all $a \in A$ and $m \in M, h \in H$.

Proof. Let us first show that $G(M)$ is an object of $\widetilde{\mathcal{H}}(\mathcal{M}_k)^H_A$. It is routine to check that $G(M)$ is a right (H, α) -Hom-comodule and a right (A, β) -Hom-module. Now we only

check the compatibility condition, for all $a \in A$. Indeed,

$$\begin{aligned}
& \rho_{G(M)}((m \otimes h) \cdot a) \\
&= \rho_{G(M)}(m \cdot a_{[0]} \otimes ha_{[1]}) \\
&= \mu^{-1}(m) \cdot \beta^{-1}(a_{[0]}) \otimes h_{(1)}a_{1} \otimes \alpha(h_{(2)}a_{[1](2)}) \\
&= \mu^{-1}(m) \cdot a_{[0][0]} \otimes h_{(1)}a_{[0][1]} \otimes \alpha(h_{(2)})a_{[1]} \\
&= (m \otimes h)_{[0]} \cdot a_{[0]} \otimes (m \otimes h)_{(1)}a_{[1]} \\
&= \rho(m \otimes c) \cdot a.
\end{aligned}$$

This is exactly what we have to show.

For an A -linear map $\varphi : (M, \mu) \rightarrow (N, \nu)$, we put

$$G(\varphi) = \varphi \otimes id_H : M \otimes H \rightarrow N \otimes H.$$

Standard computations show that $G(\varphi)$ is morphisms of right (A, β) -Hom-modules and right (H, α) -Hom-comodules. Let us describe the unit η and the counit δ of the adjunction. The unit is described by the coaction: for $M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$, we define $\eta_M : M \rightarrow M \otimes H$ as follows: for all $m \in M$,

$$\eta_M(m) = m_{[0]} \otimes m_{[1]}.$$

We can check that $\eta_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$. For any $N \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$, we define $\delta_N : N \otimes H \rightarrow N$, for all $n \in N$ and $h \in H$,

$$\delta_N(n \otimes h) = \varepsilon(h)n,$$

we can check that δ_N is (A, β) -linear. It is easy to check that $\eta_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$. We can check that η and δ defined above are all natural transformations and satisfied

$$G(\delta_N) \circ \eta_{G(N)} = I_{G(N)},$$

$$\delta_{F(M)} \circ F(\eta_M) = I_{F(M)},$$

for all $M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$ and $N \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A$.

4 Structure type theorem and injective type properties for relative Hom-Hopf modules

Definition 4.1. Let (H, α) be a Monoidal Hom Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra. The map $\phi : (H, \alpha) \rightarrow (A, \beta)$ is called a total integral such that the following conditions are satisfied:

$$\rho_A \phi = (\phi \otimes id_H) \Delta_H, \quad \phi \alpha = \beta \phi, \quad \phi(1_H) = 1_A.$$

Lemma 4.2. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra with a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$ and $M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$,

$$\lambda_M : M \otimes H \rightarrow M, \quad m \otimes h \mapsto \mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]})\alpha(h)).$$

Then the following assertions hold:

- (1) $\lambda_M \rho_M = id_M$,
- (2) λ_M is a morphism of right (H, α) -Hom-comodules, and the right (H, α) -Hom-coaction on $M \otimes H$ given by $\rho(m \otimes h) = (\mu(m) \otimes h_{(1)}) \otimes \alpha^{-1}(h_{(2)})$ for any $m \in M$ and $h \in H$,
- (3) if $\phi : (H, \alpha) \rightarrow (Z(A), \beta)$ (the center of A) is a multiplication map, then λ_M is a morphism in $\widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$.

Proof. (1) For any $m \in M$, we have

$$\begin{aligned}
\lambda_M \rho_M(m) &= \lambda_M(m_{[0]} \otimes m_{[1]}) \\
&= \mu^{-1}(m_{[0][0]}) \cdot \phi(S(m_{[0][1]})\alpha(m_{[1]})) \\
&= m_{[0]} \cdot \phi(S(m_{1})m_{[1](2)}) \\
&= m_{[0]} \cdot \phi(\varepsilon(m_{[1]})) \\
&= \mu^{-1}(m) \cdot 1_A = m.
\end{aligned}$$

(2) For any $m \in M$ and $h \in H$, we have

$$\begin{aligned}
&\rho_M \lambda_M(m \otimes h) \\
&= \rho_M(\mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]})\alpha(h))) \\
&= \mu^{-1}(m_{[0][0]}) \cdot \phi(S(m_{[1](2)})\alpha(h_{(1)})) \otimes \alpha^{-1}(m_{[0][1]})(S(m_{1})\alpha(h_{(2)})) \\
&= \mu^{-2}(m_{[0]}) \cdot \phi(\alpha(S(m_{[1](2)(2)}))\alpha(h_{(1)})) \otimes \alpha^{-1}(m_{1})(\alpha(S(m_{[1](2)(1)}))\alpha(h_{(2)})) \\
&= \mu^{-2}(m_{[0]}) \cdot \phi(S(m_{[1](2)})\alpha(h_{(1)})) \otimes m_{1(1)}(\alpha(S(m_{1(2)}))\alpha(h_{(2)})) \\
&= \mu^{-2}(m_{[0]}) \cdot \phi(S(m_{[1](2)})\alpha(h_{(1)})) \otimes (\alpha(m_{1(1)})\alpha(S(m_{1(2)})))h_{(2)} \\
&= \mu^{-2}(m_{[0]}) \cdot \phi(\alpha^{-1}(S(m_{[1]}))\alpha(h_{(1)})) \otimes \alpha(h_{(2)}) \\
&= (\lambda_M \otimes id_H)((\mu^{-1}(m) \otimes h_{(1)}) \otimes \alpha(h_{(2)})) \\
&= (\lambda_M \otimes id_H)\rho_{M \otimes H}(m \otimes h).
\end{aligned}$$

(3) For any $m \in M$, $h \in H$ and $b \in A$, we have

$$\begin{aligned}
&\lambda_M((m \otimes h) \cdot b) \\
&= \lambda_M(m \cdot b_{[0]} \otimes hb_{(1)}) \\
&= \mu^{-1}(m_{[0]} \cdot b_{[0][0]}) \cdot \phi(S(m_{[1]}b_{[0][1]})\alpha(hb_{[1]})) \\
&= \mu^{-1}(m_{[0]} \cdot b_{[0][0]}) \cdot \phi(S(m_{[1]})S(b_{[0][1]})\alpha(hb_{[1]})) \\
&= \mu^{-1}(m_{[0]} \cdot b_{[0][0]}) \cdot \phi(\alpha(S(m_{[1]})[S(b_{[0][1]})hb_{[1]}])) \\
&= \mu^{-1}(m_{[0]} \cdot b_{[0][0]}) \cdot \phi(\alpha(S(m_{[1]})[S(b_{[0][1]})(b_{[1]}h)])) \\
&= \mu^{-1}(m_{[0]} \cdot b_{[0][0]}) \cdot \phi(\alpha(S(m_{[1]})[(\alpha^{-1}(S(b_{[0][1]}))b_{[1]})\alpha(h)])) \\
&= (\mu^{-1}(m_{[0]}) \cdot b_{[0]}) \cdot \phi(\alpha(S(m_{[1]})[(\alpha^{-1}(S(b_{1}))\alpha^{-1}(b_{[1](2)}))\alpha(h)])) \\
&= (\mu^{-1}(m_{[0]}) \cdot \beta^{-1}(b)) \cdot \phi(\alpha(S(m_{[1]})\alpha^2(h))) \\
&= m_{[0]} \cdot (\beta^{-1}(b)\phi(S(m_{[1]})\alpha(h))) \\
&= m_{[0]} \cdot (\phi(\alpha^{-4}(S(m_{[1]}))\alpha^{-3}(h))\beta^{-1}(b)) \\
&= (\mu^{-1}(m_{[0]}) \cdot \phi(S(m_{[1]})\alpha(h))) \cdot b \\
&= \lambda_M(m \otimes h) \cdot b.
\end{aligned}$$

Proposition 4.3. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom comodule algebra with a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$. Then $M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$ is injective as a right (H, α) -Hom-comodule.

If H is a Hopf algebra, then we obtain the main result of [3, Theorem 1].

Corollary 4.4. Let H be a Hopf algebra and A a right H -comodule algebra. If there is a right H -comodule map $\phi : (H, \alpha) \rightarrow (A, \beta)$ such that $\phi(1_H) = 1_A$, then every relative (H, A) -Hopf-module is injective as a right H -comodule.

Let M be a relative Hom-Hopf module, and let

$$M_0 = \{m \in M \mid \rho_M(m) = \mu^{-1}(m) \otimes 1_H\}$$

be an invariant subspace of M and M_0 is a right (C, β) -Hom-module, where

$$C = \{b \in A \mid \rho_A(b) = \beta^{-1}(b) \otimes 1_H\}$$

is a subalgebra of A .

Proposition 4.5. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra with a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$ and $M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$. Assume that ϕ is a multiplication map and let

$$\tau_M : (M, \mu) \rightarrow (M, \mu)$$

be the trace map defined by

$$m \mapsto m_{[0]} \cdot \phi(S(m_{[1]})).$$

Then the following assertions hold:

- (1) $\tau_M(m) \in M_0$ and $\tau|_{M_0} = id$,
- (2) $\tau_A : (A, \beta) \rightarrow (C, \beta)$ defined by $b \mapsto b_{[0]}\phi(S(b_{[1]}))$ is a morphism of left (C, β) -Hom-modules, so that (C, β) is a direct summand of (A, β) as a sum of left (C, β) -Hom-modules,
- (3) if $Im\phi \subseteq Z(A)$, then $\tau_M : (M, \mu) \rightarrow (M, \mu)$ is a morphism of right (C, β) -Hom-modules.

The exact sequence

$$(M, \mu) \xrightarrow{\tau_M} (M_0, \mu) \longrightarrow 0,$$

thus obtained splits as a sequence of right (C, β) -Hom-modules.

Proof. (1) For any $m \in M$, we have

$$\begin{aligned} \rho(\tau_M(m)) &= \rho(m_{[0]}\phi(S(m_{[1]}))) \\ &= m_{[0][0]}\phi(S(m_{[1](2)})) \otimes m_{[0][1]}\phi(S(m_{1})) \\ &= \mu^{-1}(m_{[0]})\phi(\alpha(S(m_{[1](2)(2)}))) \otimes m_{1}\phi(\alpha(S(m_{[1](2)(1)}))) \\ &= \mu^{-1}(m_{[0]})\phi(S(m_{[1](2)})) \otimes \alpha(m_{1(1)})\phi(\alpha(S(m_{1(2)}))) \\ &= \mu^{-1}(m_{[0]})\phi(\alpha^{-1}(S(m_{[1]}))) \otimes 1_H \\ &= \mu^{-1}(\tau_M(m)) \otimes 1_H. \end{aligned}$$

For any $n \in M_0$,

$$\begin{aligned}\tau_M(n) &= n_{[0]}\phi(S(n_{[1]})) \\ &= \mu^{-1}(n)1_A = n.\end{aligned}$$

(2) For any $c \in C$ and $a \in A$,

$$\begin{aligned}\tau_A(ca) &= (c_{[0]}a_{[0]})\phi(S(c_{[1]}a_{[1]})) \\ &= (\beta^{-1}(c)a_{[0]})\phi(\alpha(S(a_{[1]}))) \\ &= c(a_{[0]} \cdot \phi(S(a_{[1]}))) = c\tau_A(a),\end{aligned}$$

thus, $\tau_A : (A, \beta) \rightarrow (C, \beta)$ is a morphism of left (C, β) -Hom-modules, and by (1), (C, β) is a direct summand of (A, β) as a sum of left (C, β) -Hom-modules.

(3) For any $c \in C$ and $m \in M$,

$$\begin{aligned}\tau_M(m \cdot c) &= (m_{[0]} \cdot c_{[0]})\phi(S(m_{[1]}c_{[1]})) \\ &= (m_{[0]} \cdot \beta^{-1}(c))\phi(\alpha(S(m_{[1]}))) \\ &= \mu(m_{[0]}) \cdot (\beta^{-1}(c))\phi(S(m_{[1]})) \\ &= \mu(m_{[0]}) \cdot (\phi(S(m_{[1]}))\beta^{-1}(c)) \\ &= (m_{[0]} \cdot \phi(S(m_{[1]}))) \cdot c = \tau_M(m) \cdot c.\end{aligned}$$

Thus, τ_M is a morphism of right (C, β) -Hom-modules, and by (1), the exact sequence

$$(M, \mu) \xrightarrow{\tau_M} (M_0, \mu) \longrightarrow 0.$$

Thus obtained splits as a sequence of right (C, β) -Hom-modules.

5 A Maschke-type theorem for relative Hom-Hopf modules

In this section, we shall give necessary and sufficient conditions for the functor F which forget the (H, α) -coaction to be separable, we prove a Maschke type theorem for relative Hom-Hopf modules as an application.

Definition 5.1. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra. A k -linear map

$$\theta : (H, \alpha) \otimes (H, \alpha) \rightarrow (A, \beta)$$

such that $\theta \circ (\alpha \otimes \alpha) = \beta \circ \theta$ is called a normalized (A, β) -integral, if θ satisfies the following conditions:

(1) For all $h, g \in H$,

$$\begin{aligned}&\theta(\alpha^{-1}(g) \otimes h_{(1)}) \otimes \alpha(h_{(2)}) \\ &= \beta(\theta(g_{(2)} \otimes \alpha^{-1}(h))_{[0]}) \otimes g_{(1)}\theta(g_{(2)} \otimes \alpha^{-1}(h))_{[1]}.\end{aligned}\tag{5. 1}$$

(2) For all $h \in H$,

$$\theta(h_{(1)} \otimes h_{(2)}) = 1_A \varepsilon(h). \quad (5.2)$$

(3) For all $a \in A, h, g \in H$,

$$\beta^2(a_{[0][0]})\theta(\alpha^{-1}(g)a_{[0][1]} \otimes \alpha^{-1}(h)\alpha^{-1}(a_{[1]})) = \theta(g \otimes h)a. \quad (5.3)$$

Theorem 5.2. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra, the following assertions are equivalent,

- (1) The left adjoint F in Proposition 3.3 is separable,
- (2) There exists a normalized (A, β) -integral $\theta : (H, \alpha) \otimes (H, \alpha) \rightarrow (A, \beta)$.

Proof. (2) \implies (1). For any relative Hom-Hopf module M , we define

$$\begin{aligned} \nu_M : M \otimes H &\rightarrow M, \\ \nu_M(m \otimes h) &= \mu(m_{[0]})\theta(m_{[1]} \otimes \alpha^{-1}(h)), \end{aligned}$$

for all $m \in M$ and $h \in H$. Now, we shall check that $\nu_M \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$. In fact, for all $m \in M, h \in H$ and $a \in A$, it is easy to get that

$$\nu_M(\mu(m) \otimes \alpha(h)) = \mu(\nu_M(m \otimes h)).$$

We also have

$$\begin{aligned} &\nu_M((m \otimes h) \cdot a) \\ &= \nu_M(ma_{[0]} \otimes ha_{[1]}) \\ &= (\mu(m_{[0]}) \cdot \beta(a_{[0][0]}))\theta(m_{[1]}a_{[0][1]} \otimes \alpha^{-1}(h)\alpha^{-1}(a_{[1]})) \\ &= \mu^2(m_{[0]}) \cdot (\beta(a_{[0][0]})\beta^{-1}(\theta(m_{[1]}a_{[0][1]} \otimes \alpha^{-1}(h)\alpha^{-1}(a_{[1]}))) \\ &= \mu^2(m_{[0]}) \cdot (\beta(a_{[0][0]})\theta(\alpha^{-1}(m_{[1]})\alpha^{-1}(a_{[0][1]}) \otimes \alpha^{-2}(h)\alpha^{-2}(a_{[1]}))) \\ &\stackrel{(5.3)}{=} \mu^2(m_{[0]}) \cdot (\theta(m_{[1]} \otimes \alpha^{-1}(h))\beta^{-1}(a)) \\ &= (\mu(m_{[0]}) \cdot \theta(m_{[1]} \otimes \alpha^{-1}(h))) \cdot a \\ &= (\nu_M(m \otimes h)) \cdot a. \end{aligned}$$

Hence it is a morphism of (A, β) -Hom-modules. Next, we shall check that ν_M is a morphism of Hom-comodules over (H, α) . It is sufficient to check that

$$\rho_M \circ \nu_M = (\nu_M \otimes id_H) \circ \rho_M$$

holds. For all $m \in M$ and $h \in H$, we have

$$\begin{aligned} &\rho_M \circ \nu_M(m \otimes h) \\ &= \rho_M(\mu(m_{[0]})\theta(m_{[1]} \otimes \alpha^{-1}(h))) \\ &= (\mu(m_{[0]})\theta(m_{(1)} \otimes \alpha^{-1}(h)))_{[0]} \otimes (\mu(m_{[0]})\theta(m_{[1]} \otimes \alpha^{-1}(h)))_{[1]} \\ &= \mu(m_{[0][0]})\theta(m_{[1]} \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(m_{[0][1]})\theta(m_{(1)} \otimes \alpha^{-1}(h))_{[1]} \\ &= m_{[0]}\theta(\alpha(m_{[1](2)}) \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(m_{1})\theta(\alpha(m_{[1](2)}) \otimes \alpha^{-1}(h))_{[1]} \\ &\stackrel{(5.1)}{=} m_{[0]}\beta^{-1}(\theta(m_{[1]} \otimes h_{(1)})) \otimes \alpha(h_{(2)}) \\ &= m_{[0]}\theta(\alpha^{-1}(m_{[1]}) \otimes \alpha^{-1}(h_{(1)})) \otimes \alpha(h_{(2)}) \\ &= (\nu_M \otimes id_H) \circ \rho_M(m \otimes h). \end{aligned}$$

For all $m \in M$, since

$$\begin{aligned}\nu_M \circ \eta_M(m) &= \nu_M(m_{[0]} \otimes m_{[1]}) \\ &= \mu(m_{[0][0]})\theta(m_{[0][1]} \otimes \alpha^{-1}(m_{[1]})) \\ &= m_{[0]}\theta(m_{1} \otimes m_{[1](2)}) \stackrel{(5.2)}{=} m.\end{aligned}$$

So the left adjoint F in Proposition 3.3 is separable follows by Rafael theorem.

(1) \implies (2). We consider the following relative Hom-Hopf module $A \otimes H$, and the (A, β) -actions and (H, α) -coaction are defined as follows:

$$\begin{cases} (a \otimes h) \cdot b = ab_{[0]} \otimes hb_{[1]}; \\ \rho_{A \otimes H}(a \otimes h) = (\beta^{-1}(a) \otimes h_{(1)}) \otimes \alpha(h_{(2)}), \end{cases}$$

for any $a, b \in A$ and $h \in H$.

Evaluating at this object, the retraction ν of the unit of the adjunction in Proposition 3.3 yields a morphism

$$\nu_{A \otimes H} : (A \otimes H) \otimes H \rightarrow A \otimes H$$

such that, for all $a \in A, h \in H$,

$$\nu_{A \otimes H}((a \otimes h_{(1)}) \otimes h_{(2)}) = a \otimes h.$$

It can be used to construct θ as follows:

$$\theta : H \otimes H \rightarrow A,$$

$$\theta(h \otimes g) = r_A(id_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes h) \otimes g),$$

where r means the right unit constraint. For all $h \in H$, since

$$\begin{aligned}\theta(h_{(1)} \otimes h_{(2)}) &= r_A(id_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes h_{(1)}) \otimes h_{(2)}) \\ &= r_A(id_A \otimes \varepsilon)(1_A \otimes h) = 1_A \varepsilon(h).\end{aligned}$$

Hence condition Eq.(5.2) follows. It can be seen to obey Eq.(5.3) by naturality and the (A, β) -modules map of ν .

The verification of Eq.(5.1) is more involved. For any right (H, α) -Hom-comodule M , we consider the relative Hom-Hopf module $M \otimes A$, the (A, β) -action and (H, α) -coaction are defined as follows: for all $m \in M$ and $a, b \in A$,

$$\begin{cases} (m \otimes a) \cdot b = \mu^{-1}(m) \otimes a\beta(b), \\ \rho_{M \otimes A}(m \otimes a) = (m_{[0]} \otimes a_{[0]}) \otimes m_{[1]}a_{[1]}. \end{cases}$$

In particular, there is a relative Hom-Hopf module $H \otimes A$ and the map

$$\xi : H \otimes A \rightarrow A \otimes H$$

$$\xi(h \otimes a) = \beta(a_{[0]}) \otimes \alpha^{-1}(h)a_{[1]}.$$

Since ξ is both right (A, β) -linear and right (H, α) -colinear, thus we have

$$\begin{aligned}\xi(\nu_{H \otimes A}((h \otimes a) \otimes g)) &= \nu_{A \otimes H}((\xi \otimes id_H)((h \otimes a) \otimes g)) \\ &= \nu_{A \otimes H}((\beta(a_{[0]}) \otimes \alpha^{-1}(h)a_{[1]}) \otimes g).\end{aligned}\quad (5.4)$$

It is not hard to check that $GF(H \otimes A) = (H \otimes A) \otimes H \in {}^H \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$, and its left (H, α) -Hom comodule structure is given by

$$(h \otimes a) \otimes g \mapsto \alpha(h_{(1)}) \otimes ((h_{(2)} \otimes \beta^{-1}(a)) \otimes \alpha^{-1}(g)).$$

Also $H \otimes A \in {}^H \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$, and the left (H, α) -coaction of $H \otimes A$ is given by

$$h \otimes a \mapsto \alpha(h_{(1)}) \otimes (h_{(2)} \otimes \beta^{-1}(a)).$$

We also get $\nu_{H \otimes A} : (H \otimes A) \otimes H \rightarrow H \otimes A$ is a Hom morphism in ${}^H \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$, which means

$$\begin{aligned}\nu_{H \otimes A}((h \otimes a) \otimes g)_{[-1]} \otimes \nu_{H \otimes A}((h \otimes a) \otimes g)_{[0]} \\ = \alpha(h_{(1)}) \otimes \nu_{H \otimes A}((h_{(2)} \otimes \beta^{-1}(a)) \otimes \alpha^{-1}(g)).\end{aligned}\quad (5.5)$$

Thus we conclude that $\nu_{H \otimes A}$ is left and right (H, α) -colinear. Take $h, g \in H$, and put

$$\nu_{A \otimes H}((1_A \otimes h) \otimes g) = \sum_i a_i \otimes q_i \in A \otimes H,$$

$$\nu_{H \otimes A}((h \otimes 1_A) \otimes g) = \sum_i p_i \otimes b_i \in H \otimes A,$$

we obtain

$$\begin{aligned}& \beta(\theta(h_{(2)} \otimes \alpha^{-1}(g))_{[0]}) \otimes h_{(1)} \theta(h_{(2)} \otimes \alpha^{-1}(g))_{[1]} \\ &= \beta(r_A(id_A \otimes \varepsilon) \nu_{A \otimes H}((1_A \otimes h_{(2)}) \otimes \alpha^{-1}(g))_{[0]}) \otimes h_{(1)} \\ & \quad \cdot (r_A(id_A \otimes \varepsilon) \nu_{A \otimes H}((1_A \otimes h_{(2)}) \otimes \alpha^{-1}(g))_{[1]}) \\ &\stackrel{(5.4)}{=} \beta(r_A(id_A \otimes \varepsilon) \xi \nu_{H \otimes A}((h_{(2)} \otimes 1_A) \otimes \alpha^{-1}(g))_{[0]}) \otimes h_{(1)} \\ & \quad \cdot (r_A(id_A \otimes \varepsilon) \xi \nu_{H \otimes A}((h_{(2)} \otimes 1_A) \otimes \alpha^{-1}(g))_{[1]}) \\ &\stackrel{(5.5)}{=} \sum_i \beta(r_A(id_A \otimes \varepsilon) \xi(p_{i(2)} \otimes \beta^{-1}(b_i))_{[0]}) \otimes p_{i(1)} (r_A(id_A \otimes \varepsilon) \xi(p_{i(2)} \otimes \beta^{-1}(b_i))_{[1]}) \\ &= \sum_i \beta(r_A(id_A \otimes \varepsilon)(b_{i[0]} \otimes \alpha^{-1}(p_{i(2)})b_{i[1]})_{[0]}) \otimes p_{i(1)} (r_A(id_A \otimes \varepsilon)(b_{i[0]} \otimes \alpha^{-1}(p_{i(2)})b_{i[1]})_{[1]}) \\ &= \sum_i \beta(b_{i[0]}) \otimes p_{i(1)} \varepsilon(p_{i(2)})(b_{i[1]}) \\ &= \sum_i \xi(p_i \otimes b_i) = \xi(\nu_{H \otimes A}((h \otimes 1_A) \otimes g)).\end{aligned}$$

Use the fact that $\nu_{A \otimes H}$ is a morphism of right (H, α) -Hom comodules, we also have

$$\begin{aligned}
& \theta(\alpha^{-1}(h) \otimes g_{(1)}) \otimes \alpha(g_{(2)}) \\
&= r_A(id_A \otimes \varepsilon)\nu_{A \otimes H}((1_A \otimes \alpha^{-1}(h)) \otimes g_{(1)}) \otimes \alpha(g_{(2)}) \\
&= \sum_i r_A(id_A \otimes \varepsilon)(\beta^{-1}(a_i) \otimes q_{i(1)}) \otimes \alpha(q_{i(2)}) \\
&= \sum_i a_i \otimes q_i = \nu_{A \otimes H}((1_A \otimes h) \otimes g) \\
&\stackrel{(5.4)}{=} \xi(\nu_{H \otimes A}((h \otimes 1_A) \otimes g)).
\end{aligned}$$

Hence, we can get condition Eq.(5.1).

We will now investigate the relation between the total integrals and the normalized (A, β) -integrals. This will explain our terminology, and we will also prove that the forgetful functor is separable if and only if there exists a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$ such that the image of $\rho_A \circ \phi$ is contained in the center of $H \otimes A$.

Proposition 5.3. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra. If $\theta : (H, \alpha) \otimes (H, \alpha) \rightarrow (A, \beta)$ is a normalized (A, β) -integral for (H, A, H) , then the k -linear map

$$\phi : (H, \alpha) \rightarrow (A, \beta), \quad \phi(h) = \theta(1_H \otimes h),$$

for all $h \in H$, is a total integral.

Proof. Notice first that $\phi(1_H) = \theta(1_H \otimes 1_H) = \varepsilon_H(1_H)1_A = 1_A$. Since

$$\begin{aligned}
& \theta(\alpha^{-1}(g) \otimes \alpha^{-1}(h_{(1)})) \otimes \alpha(h_{(2)}) \\
&= (\theta(\alpha(g_{(2)}) \otimes \alpha^{-1}(h)))_{(0)} \otimes \alpha(g_{(1)})(\theta(\alpha(g_{(2)}) \otimes \alpha^{-1}(h)))_{(1)}.
\end{aligned}$$

It follows by taking $g = 1_H$ that

$$\theta(1_H \otimes \alpha^{-1}(h_1)) \otimes \alpha(h_2) = \theta(1_H \otimes \alpha^{-1}(h))_{[0]} \otimes \alpha(\theta(1_H \otimes \alpha^{-1}(h))_{[1]}),$$

and applying $\alpha \otimes \alpha^{-1}$ to the above identity, we have

$$\theta(1_H \otimes \alpha^{-1}(h_1)) \otimes h_2 = \theta(1_H \otimes \alpha^{-1}(h))_{[0]} \otimes \theta(1_H \otimes \alpha^{-1}(h))_{[1]}.$$

So we obtain

$$\phi(h_1) \otimes h_2 = \phi(h)_{[0]} \otimes \phi(h)_{[1]}.$$

It is easy to check that $\phi\alpha = \beta\phi$. So ϕ is a total integral.

Let $\phi : (H, \alpha) \rightarrow (A, \beta)$ be a total integral for the right (H, α) -Hom-comodule algebra (A, β) , and define

$$\theta : (H, \alpha) \otimes (H, \alpha) \rightarrow (A, \beta), \quad \theta(g \otimes h) = \phi(hS^{-1}(g)),$$

for all $g, h \in H$.

Theorem 5.4. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra, and $\phi : (H, \alpha) \rightarrow (A, \beta)$ a total integral. If

$$g\phi(h)_{[1]} \otimes \phi(h)_{[0]} = \phi(h)_{[1]}g \otimes \phi(h)_{[0]}, \quad \phi(h) \in Z(A).$$

Then θ is a normalized (A, β) -integral.

Proof. For any $h, g \in H$ and $a \in A$, we have

$$\begin{aligned} & \beta^2(a_{[0][0]})\theta(\alpha^{-1}(g)a_{[0][1]} \otimes \alpha^{-1}(h)\alpha^{-1}(a_{[1]})) \\ = & \beta(a_{[0]})\theta(\alpha^{-1}(g)a_{1} \otimes \alpha^{-1}(h)a_{[1](2)}) \\ = & \beta(a_{[0]})\phi(\alpha^{-1}(h)a_{[1](2)}S^{-1}(\alpha^{-1}(g)a_{1})) \\ = & \beta(a_{[0]})\phi(h[(\alpha^{-1}(a_{[1](2)}S^{-1}(\alpha^{-1}(a_{1})))S^{-1}(\alpha^{-1}(g))]) \\ = & a\phi(hS^{-1}(g)) \\ = & \theta(g \otimes h)a, \end{aligned}$$

and

$$\begin{aligned} & \beta(\theta(g_{(2)} \otimes \alpha^{-1}(h))_{[0]}) \otimes g_{(1)}\theta(g_{(2)} \otimes \alpha^{-1}(h))_{[1]} \\ = & \beta(\phi(\alpha^{-1}(h)S^{-1}(g_{(2)}))_{[0]}) \otimes \phi(\alpha^{-1}(h)S^{-1}(g_{(2)}))_{[1]}g_{(1)} \\ = & \phi(h_{(1)}S^{-1}(\alpha(g_{(2)(2)}))) \otimes (\alpha^{-1}(h_{(2)})S^{-1}(g_{(2)(1)}))g_{(1)} \\ = & \phi(h_{(1)}S^{-1}(g_{(2)})) \otimes (\alpha^{-1}(h_{(2)})S^{-1}(g_{(1)(2)}))\alpha(g_{(1)(1)}) \\ = & \phi(h_{(1)}S^{-1}(g_{(2)})) \otimes h_{(2)}(S^{-1}(g_{(1)(2)})g_{(1)(1)}) \\ = & \phi(h_{(1)}S^{-1}(\alpha^{-1}(g))) \otimes \alpha(h_{(2)}) \\ = & \theta(\alpha^{-1}(g) \otimes h_{(1)}) \otimes \alpha(h_{(2)}), \end{aligned}$$

$$\theta(h_1 \otimes h_2) = \varphi(h_2S^{-1}(h_1)) = \varepsilon_H(h)1_A.$$

It is easy to check that $\phi\alpha = \beta\phi$. So θ is a normalized (A, β) -integral.

Since separable functors reflect well the semisimplicity of the objects of a category, by Theorem 5.2, we will prove a Maschke type theorem for relative Hom-Hopf modules as an application.

Lemma 5.5. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra with a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$ and $(M, \mu), (N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)_A^H$ and a Hom-morphism $f : (N, \nu) \rightarrow (M, \mu)$. Let

$$f_\phi : N \xrightarrow{\rho_N} N \otimes H \xrightarrow{f \otimes id_H} M \otimes H \xrightarrow{\tau} M,$$

that is,

$$f_\phi(n) = \mu^{-1}(f(n_{[0]})) \cdot \phi(S(f(n_{[0]}))_{[1]}\alpha(n_{[1]})),$$

for any $n \in N$. Then the following assertions hold:

- (1) f_ϕ is a morphism of right (H, α) -Hom-comodules,
- (2) if $f : (N, \nu) \rightarrow (M, \mu)$ is a morphism of right (A, β) -Hom-modules and $\phi : (H, \alpha) \rightarrow (Z(A), \beta)$ is a multiplication map, then f_ϕ is a morphism of right (A, β) -Hom-module.

Proof. (1) For any $n \in N$, we have

$$\begin{aligned}
\rho_M(f_\phi(n)) &= \rho_M(\mu^{-1}(f(n_{[0]})_{[0]}) \cdot \phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]}))) \\
&= \mu^{-1}(f(n_{[0]})_{[0][0]}) \cdot \phi(S(f(n_{[0]})_{[1](2)})\alpha(n_{1})) \\
&\quad \otimes \alpha^{-1}(f(n_{[0]})_{[0][1]}) (S(f(n_{[0]})_{1})\alpha(n_{[1](2)})) \\
&= \mu^{-2}(f(n_{[0]})_{[0]}) \cdot \phi(\alpha(S(f(n_{[0]})_{[1](2)(2)}))\alpha(n_{1})) \\
&\quad \otimes \alpha^{-1}(f(n_{[0]})_{1}) (\alpha(S(f(n_{[0]})_{[1](2)(1)}))\alpha(n_{[1](2)})) \\
&= \mu^{-2}(f(n_{[0]})_{[0]}) \cdot \phi(S(f(n_{[0]})_{[1](2)})\alpha(n_{1})) \\
&\quad \otimes f(n_{[0]})_{1(1)} (\alpha(S(f(n_{[0]})_{1(2)}))\alpha(n_{[1](2)})) \\
&= \mu^{-2}(f(n_{[0]})_{[0]}) \cdot \phi(S(f(n_{[0]})_{[1](2)})\alpha(n_{1})) \\
&\quad \otimes (\alpha(f(n_{[0]})_{1(1)})\alpha(S(f(n_{[0]})_{1(2)}))n_{[1](2)}) \\
&= \mu^{-2}(f(n_{[0]})_{[0]}) \cdot \phi(\alpha^{-1}(S(f(n_{[0]})_{[1]}))\alpha(n_{1})) \otimes \alpha(n_{[1](2)}) \\
&= \mu^{-1}(f(n_{[0]})_{[0]}) \cdot \phi(S(f(n_{[0]})_{[1]})\alpha(n_{[0][1]})) \otimes n_{[1]} \\
&= (f_\phi \otimes id_H)\rho_N(n).
\end{aligned}$$

(2) For any $n \in N$ and $b \in A$, we have

$$\begin{aligned}
&f_\phi(n \cdot b) \\
&= \mu^{-1}(f(n_{[0]})_{[0]} \cdot b_{[0][0]}) \cdot \phi(S(f(n_{[0]})_{[1]})b_{[0][1]})\alpha(n_{[1]}b_{[1]}) \\
&= \mu^{-1}(f(n_{[0]})_{[0]} \cdot b_{[0][0]}) \cdot \phi([S(f(n_{[0]})_{[1]})b_{[0][1]}][\alpha(b_{[1]})\alpha(n_{[1]})]) \\
&= \mu^{-1}(f(n_{[0]})_{[0]} \cdot b_{[0][0]}) \cdot \phi(\alpha(S(f(n_{[0]})_{[1]})[S(b_{[0][1]})b_{[1]}n_{[1]}]) \\
&= \mu^{-1}(f(n_{[0]})_{[0]} \cdot b_{[0][0]}) \cdot \phi(\alpha(S(f(n_{[0]})_{[1]}))[\alpha^{-1}(S(b_{[0][1]}))b_{[1]})\alpha(n_{[1]})]) \\
&= (\mu^{-1}(f(n_{[0]})_{[0]}) \cdot b_{[0]}) \cdot \phi(\alpha(S(f(n_{[0]})_{[1]}))[(\alpha^{-1}(S(b_{1}))\alpha^{-1}(b_{[1](2)}))\alpha(n_{[1]})]) \\
&= (\mu^{-1}(f(n_{[0]})_{[0]}) \cdot \beta^{-1}(b)) \cdot \phi(\alpha(S(f(n_{[0]})_{[1]}))\alpha^2(n_{[1]})) \\
&= f(n_{[0]})_{[0]} \cdot (\beta^{-1}(b)\phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]}))) \\
&= f(n_{[0]})_{[0]} \cdot (\phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]}))\beta^{-1}(b)) \\
&= (\mu^{-1}(f(n_{[0]})_{[0]}) \cdot \phi(S(f(n_{[0]})_{[1]})\alpha(n_{[1]}))) \cdot b = f_\phi(n) \cdot b.
\end{aligned}$$

Theorem 5.6. Let (H, α) be a monoidal Hom-Hopf algebra and (A, β) a right (H, α) -Hom-comodule algebra with a total integral $\phi : (H, \alpha) \rightarrow (A, \beta)$. If $\phi : (H, \alpha) \rightarrow (Z(A), \beta)$ is a multiplication map, then every short exact sequence of relative Hom-Hopf modules

$$0 \longrightarrow (M, \mu) \xrightarrow{f} (N, \nu) \xrightarrow{g} (P, \pi) \longrightarrow 0$$

which splits as a sequence of (A, β) -Hom-modules also splits as a sequence of relative Hom-Hopf modules.

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